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On Generalized Hexagons and a Near Octagon whose Lines have Three Points

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Proofs are given of the facts that any finite generalized hexagon of order $(2, t)$ is isomorphic to the classical generalized hexagon associated with the group $G_2(2)$ or to its dual if $t=2$ and that it is isomorphic to the classical generalized hexagon associated with the group ${}^3D_4(2)$ if $t=8$. Furthermore, it is shown that any near octagon of order $(2, 4; 0, 3)$ is isomorphic to the known one associated with the sporadic simple group HJ.

1. INTRODUCTION

In [7], one of us stated the uniqueness up to isomorphism and duality of the generalized hexagon of order $(2, 2)$ (i.e. each line has three points and each point belongs to three lines). The proof remained unpublished but was recently communicated to the other author who observed that the same general idea, suitably adapted, also yields the uniqueness of the generalized hexagon of order $(2, 8)$ and that of the near octagon of order $(2, 4; 0, 3)$ (see definitions below). This paper presents those three results.

It should be mentioned that, meanwhile, F. Timmesfeld ([6, (3.4)]) has outlined a characterization by group-theoretical means of the generalized hexagon of order $(2, 8)$ and of the dual of the classical hexagon of order $(2, 2)$.

By a result of W. Haemers and C. Roos [3], the only possible orders of a finite generalized hexagon with lines of length three are $(2, 1)$, $(2, 2)$ and $(2, 8)$. It is easy to see that there is a unique generalized hexagon of order $(2, 1)$ and that it admits $GL(3, 2)$ as a group of automorphisms. We shall not discuss this case any further.

2. DEFINITIONS AND STATEMENT OF RESULTS

Graphs are undirected, without loops or multiple edges. Subsets of the point set of a graph are often identified with their induced subgraphs. For γ a point of the graph Γ and a nonnegative integer i, the set of points in Γ at distance i from γ is denoted by $\Gamma_i(\gamma)$. An *i-path* is a path of length i.

An *incidence system* $(\mathcal{P}, \mathcal{L})$ is a set of *points* $\mathcal{P}$ and a collection $\mathcal{L}$ of subsets of $\mathcal{P}$ whose elements are called *lines*. To such a system we associate the *collinearity graph* Γ whose vertices are the points and in which adjacency for two distinct vertices is collinearity.

The following notion appears in [5]. *A regular near* $2d$*-gon of order* $(s, t; t_2, \ldots, t_{d-1})$ *and of diameter* d is an incidence system $(\mathcal{P}, \mathcal{L})$ such that each line contains exactly $s+1$ points, each point is on exactly $t+1$ lines, the following holds with $t_1 = 0$ and $t_d = t$: for any two points $\alpha, \beta \in \mathcal{P}$ with $\alpha \in \Gamma_i(\beta)$ there are precisely $1 + t_i$ lines through α bearing a single point of $\Gamma_{i-1}(\beta)$ while the other $t - t_i$ lines through α have no points in $\Gamma_{i-1}(\beta)$ or $\Gamma_i(\beta)$ but α (here, $0 \leq i \leq d$), and the collinearity graph Γ is connected (of diameter d automatically). *A generalized* $2d$*-gon of order* (s, t) is a regular near $2d$-gon of order $(s, t; 0, 0, \ldots, 0)$ and of diameter d.

The *dual* of a generalized $2d$-gon $(\mathcal{P}, \mathcal{L})$ is the incidence system $(\mathcal{L}, \mathcal{P}')$ where $\mathcal{P}'$ is the collection of subsets of $\mathcal{L}$ consisting of all members of $\mathcal{L}$ that have a point of $\mathcal{P}$ in common. It is easily verified that the dual of a generalized $2d$-gon of order (s, t) is a generalized $2d$-gon of order (t, s).

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THEOREM 1. *Up to isomorphism there are exactly two generalized hexagons of order* $(2,2)$. *Each of them is the dual of the other.*

THEOREM 2. *Up to isomorphism there is exactly one generalized hexagon of order* $(2,8)$.

THEOREM 3. *Up to isomorphism there is exactly one near octagon of order* $(2,4;0,3)$.

The generalized hexagons of Theorem 1 are associated with the group $G_2(2)$, and the one of Theorem 2 is associated with ${}^3D_4(2)$; see [7] for a description. The near octagon is described in [2]; it is associated with the sporadic group HJ of Hall-Janko.

The collinearity graph Γ of a regular near $2d$-gon of order and diameter as above is a *distance-regular graph with intersection array* $\{s(t+1), s(t-t_1), \ldots, s(t-t_{d-1}); 1, 1+t_2, \ldots, 1+t_{d-1}, 1+t\}$ according to the definition in [1]. This means that for any i $(0 \leqslant i \leqslant d)$ and any two points $\alpha, \beta \in \Gamma$ at mutual distance i, there are exactly $1+t_i$ points in $\Gamma_{i-1}(\alpha) \cap \Gamma_1(\beta)$ and exactly $s(t-t_i)$ points in $\Gamma_{i+1}(\alpha) \cap \Gamma_1(\beta)$ (here, $t_0 = -1$ and, as above, $t_1 = 0$).

We shall restrict attention to the case of regular near $2d$-gons with lines of size 3 (i.e. $s=2$). It is easy to see that the above incidence system is completely determined by its collinearity graph if $t_2 = 0$. Therefore, if we denote by $\kappa(i)$ the number of isomorphism classes of distance-regular graphs with intersection array i, the above theorems (without the duality statement, that is) can alternately be expressed by the following equalities:

$$\kappa(6,4,4;1,1,3)=2, \qquad \kappa(18,16,16;1,1,9)=1, \qquad \kappa(10,8,8,2;1,1,4,5)=1.$$

In a distance-regular graph Γ whose maximal cliques have size 3, *lines* are by definition maximal cliques. Note that this definition coincides with the one given above for any near $2d$-gon whose collinearity graph is Γ.

Suppose Γ is the collinearity graph of a near $2d$-gon of order $(2, t; 0, t_3, \ldots, t_{d-1})$ with $t_3 > 0$ (remember that if $d=3$, $t_3 = t$ by definition). If γ, δ are two collinear points of Γ, we shall denote by $\gamma\delta$ the line through γ, δ and by $\gamma * \delta$ the third point of $\gamma\delta$. If γ, δ are points at mutual distance 3, then there are distinct $\gamma_i \in \Gamma_1(\gamma) \cap \Gamma_2(\delta)$ and $\delta_i \in \Gamma_1(\delta) \cap \Gamma_1(\gamma_i)$ $(i = 1, 2, \ldots, t_3+1)$. Denote by $\widetilde{\gamma\delta}$ the intersection of $\Gamma_1(\gamma_i * \delta_i)$ over all i $(1 \leqslant i \leqslant t_3+1)$. Clearly $|\widetilde{\gamma\delta}| \leqslant 1$. If $|\widetilde{\gamma\delta}| = 1$ for each pair γ, δ with $\gamma \in \Gamma_3(\delta)$, we say that Γ satisfies the *regulus condition*. This notion is taken from [4].

3. Reconstruction of Γ from the Geometry of Lines Having no Point Collinear with a Given Point

In this section, $(\mathcal{P}, \mathcal{L})$ is a (nonnecessarily finite) incidence system with collinearity graph Γ. For $\omega \in \mathcal{P}$ and $n \in \mathbf{N}$, we denote by $\Gamma_{\geqslant n}(\omega)$ [resp. $\Gamma_{\leqslant n}(\omega)$] the union of all $\Gamma_i(\omega)$ over $i \geqslant n$ (resp. $i \leqslant n$), and by $Y_{\geqslant n}(\mathcal{P}, \mathcal{L}; \omega)$, or simply $Y_{\geqslant n}(\omega)$ [resp. $Y_{\leqslant n}(\omega)$], the incidence system consisting of that set and the lines (elements of $\mathcal{L}$) entirely contained in it. Our purpose is to show that, under certain conditions which are satisfied in the cases we are interested in, the system $(\mathcal{P}, \mathcal{L})$ can be recovered from the subsystem $Y_{\geqslant 2}(\omega)$. The conditions in question are the following:

(H0) Every point belongs to at least three lines and every line has at least three points.

(H1) There is no n-gon (i.e. system of n distinct points p_i and n distinct lines L_i, with $i \in \mathbb{Z}/n\mathbb{Z}$, such that $L_i \cap L_{i+1} = \{p_i\}$ for all i) for $n < 6$, but any two points at distance 2 are vertices of a hexagon.

(H2) If $a \in \mathcal{P}$, every line having at least two points in $\Gamma_3(a)$ meets $\Gamma_2(a)$ [and hence is contained in $\Gamma_{\leqslant 3}(a)$].

It is clear that the conditions are satisfied by thick generalized hexagons and by regular near $2d$-gons of order $(s, t; 0, t_3, \ldots, t_{d-1})$ and diameter d if $t_3 \neq 0$, $s \geqslant 2$ and $t \geqslant 2$.

LEMMA 1. *Let $i=1$ or 2. If the conditions* (H0), (H1), (H2) *are satisfied, and if $\omega \in \mathcal{P}$, then the incidence system $(\mathcal{P}, \mathcal{L})$ is uniquely determined by the subsystem $Y_{\geqslant i}(\omega)$. More precisely, if $(\mathcal{P}', \mathcal{L}')$ is another incidence system satisfying the conditions* (H0), (H1), (H2) *and if $\omega' \in \mathcal{P}'$, then every isomorphism $Y_{\geqslant i}(\mathcal{P}, \mathcal{L}; \omega) \to Y_{\geqslant i}(\mathcal{P}', \mathcal{L}'; \omega')$ extends uniquely to an isomorphism $(\mathcal{P}, \mathcal{L}) \to (\mathcal{P}', \mathcal{L}')$, and the latter maps ω onto ω'.*

PROOF. Let D (resp. D') denote the distance in the graph of $Y_{\geqslant i}(\omega)$ [resp. $Y_{\leqslant i}(\omega)$]; we first show that the following properties are equivalent:
(P1) $a, b \in \Gamma_i(\omega)$ and $D'(a, b)=2$;
(P2) $a, b \in \Gamma_{\geqslant i}(\omega)$, $D(a, b)=4$ and there is a line L of $Y_{\geqslant i}(\omega)$ containing a and such that $x \in L-\{a\}$ implies $D(b, x)=3$.

Suppose that (P1) holds and consider a hexagon having a and b as vertices [cf. (H1)]. Let L, L' be the sides of that hexagon which are contained in $Y_{\geqslant i}(\omega)$ and contain respectively a and b. By (H1) and (H2), every point of $L-\{a\}$ is at distance 2 of L', hence at distance 3 of b, inside $Y_{\geqslant i}(\omega)$, and (P2) follows. Conversely, if (P2) holds, it follows from (H0) and (H2) that the distance of a and b in Γ is 2, which, in view of the relation $D(a, b)=4$, is only possible if (P1) holds.

Now, we see that the set $\Gamma_i(\omega)$ precisely consists of those elements a of $\Gamma_{\geqslant i}(\omega)$ for which there exists b such that (P2) holds. Let then R denote the equivalence relation in $\Gamma_i(\omega)$ generated by all pairs (a, b) satisfying (P2). From the equivalence of (P1) and (P2), it follows that the map $p \mapsto C_p = \Gamma_1(p) \cap \Gamma_i(\omega)$, with $p \in \Gamma_{i-1}(\omega)$, is a bijection of $\Gamma_{i-1}(\omega)$ onto the set of equivalence classes of R, and that two points a, b of $\Gamma_i(\omega)$ are on the same line containing p if and only if $a, b \in C_p$ and (a, b) does not satisfy (P2). Thus, we have reconstructed $Y_{\geqslant i-1}(\omega)$. If $i=1$, we are through. If $i=2$, we use induction.

4. AUXILIARY RESULTS ON COVERS OF GRAPHS

Given two graphs Γ, Δ, we call a map $f: \Gamma \to \Delta$ sending points to points and edges to edges a *cover* of Δ whenever its restriction to $\Gamma_1(\gamma)$ is a bijection between the points of $\Gamma_1(\gamma)$ and the points of $\Delta_1(f(\gamma))$ for each point γ of Γ. Note that the cardinality of $f^{-1}(\gamma)$ only depends on the connected component of γ. We call f an *m-cover* of Δ if $|f^{-1}(\gamma)|=m$ for each point γ of Γ. Two covers $f_1: \Gamma_1 \to \Delta_1$ and $f_2: \Gamma_2 \to \Delta_2$ are called isomorphic if there are graph isomorphisms $\phi: \Gamma_1 \to \Gamma_2$ and $\psi: \Delta_1 \to \Delta_2$ such that $f_2\phi = \psi f_1$. Uniqueness of covers is always meant up to isomorphism. We shall often refer to Γ as a cover of Δ when in fact we have a map $f: \Gamma \to \Delta$ in mind.

'The' *fundamental group* of a connected graph is the group of 'homotopy classes' of closed paths with a given origin, two such paths being 'homotopic' if each one of them can be deduced from the other by successive insertions or deletions of an oriented edge followed by its inverse. Each homotopy class contains a unique *reduced* path (path of minimum length in the class). In the sequel, we shall often use the word 'path' instead of 'reduced path'. The fundamental group made abelian is canonically isomorphic with the first homology group H_1 of the graph, i.e. the group of 1-cycles (linear combinations of oriented edges with zero boundary).

Denote by $H(n, 2)$ the n-cube over $\mathbf{F}_2$, i.e. the graph whose points are the vectors in $\mathbf{F}_2^n$ and where two points are adjacent if they differ in exactly one coordinate. A closed path of length 6 in $H(n, 2)$ is called an *aperiodic hexagon* if it circumscribes two adjacent squares.

LEMMA 2. *Let n be an integer $\geqslant 2$.*
(a) *The graph $H(n, 2)$ has a unique 2-cover $\tilde{H}(n, 2)$ without 4-circuits. That cover is connected.*

(b) *The fundamental group* $\pi_1(\tilde{H}(n,2))$ *of* $\tilde{H}(n,2)$ *is generated by closed paths of* $H(n,2)$ *of the form* ghg^{-1} *with* h *an aperiodic hexagon.*
(c) *The graph* $\tilde{H}(n,2)$ *is bipartite. The involution* τ *which exchanges the pairs of points having the same images in* $H(n,2)$ *is an isomorphism. If* $\gamma \in \tilde{H}(n,2)$, *the points* γ *and* $\tau\gamma$ *are the only points at distance two from all the points which are at distance two from* γ; *thus,* τ *is central in the full automorphism group of* $\tilde{H}(n,2)$.
(d) *If* $\gamma, \delta \in \tilde{H}(n,2)$ *are such that their images are opposite vertices of* $H(n,2)$, *the group of all automorphisms of* $\tilde{H}(n,2)$ *fixing* γ *(and hence* $\tau\gamma$*) is a symmetric group on* n *letters whose even (resp. odd) elements fix (resp. exchange)* δ *and* $\tau\delta$.

PROOF. (a) Let F be the group $\langle r_1, r_2, \ldots, r_n \mid r_i^2 = 1\rangle$. Choose an origin o in $H(n,2)$. Then the paths on $H(n,2)$ with origin o are naturally labelled by the elements of F. The group $\pi = \pi_1(H(n,2))$ of closed paths is the derived group F' of F (just observe that the elementary abelian group F/F' of order 2^n is naturally bijected onto the set of vertices of $H(n,2)$). The first homology group $H_1(H(n,2)) = F'/F''$ is obviously generated by 4-circuits. Since the sum of an odd number of 4-circuits in $H_1(H(n,2))$ cannot be zero, this homology group has a unique subgroup of index 2 containing no 4-circuits. But the subgroups of index 2 of π correspond bijectively to those of $H_1(H(n,2))$ for any such subgroup of F' contains the commutator subgroup F''. Thus π has a unique subgroup of index 2 without 4-circuits. This, however, amounts to saying that $H(n,2)$ has a unique connected 2-cover $\tilde{H}(n,2)$ without 4-circuits. As any nonconnected 2-cover consists of two distinct copies of $H(n,2)$ and therefore contains 4-circuits, (a) is proved.

(b) The fundamental group π is the kernel of the natural homomorphism of F onto F_1, where

$$F_1 = \langle r_1, r_2, \ldots, r_n \mid r_i^2 = 1,\ r_i r_j = r_j r_i \text{ for all distinct } i, j\rangle,$$

and $\tilde{\pi} = \pi_1(\tilde{H}(n,2))$ is the kernel of the natural homomorphism of F onto F_2, where

$$F_2 = \langle r_1, r_2, \ldots, r_n, \varepsilon \mid r_i^2 = \varepsilon^2,\ r_i\varepsilon = \varepsilon r_i,\ r_i r_j = r_j r_i \varepsilon \text{ for all distinct } i, j\rangle.$$

Since

$$\langle r_1, r_2, \ldots, r_n \mid r_i^2 = 1,\ r_i r_j r_k = r_k r_i r_j \text{ for all distinct } i, j, k\rangle$$

is clearly another presentation of F_2 on the 'same generators' $r_1, r_2, \ldots, r_n$ as F_1, the group $\tilde{\pi}$ is generated by all elements of the form ghg^{-1} for g in F and h of the form $r_i r_j r_k r_j r_i r_k$ with i, j, k distinct. This means that h is an aperiodic hexagon, whence (b).

(c) This easily follows from the above.

(d) Taking γ as an origin in $\tilde{H}(n,2)$, the latter can be described as the graph with vertex set $F/\tilde{\pi}$ whose edges are all pairs $\{x\tilde{\pi}, y\tilde{\pi}\}$ with $x \in r_i y\tilde{\pi}$ for some i $(1 \leqslant i \leqslant n)$. Any permutation of n letters acting on F by permuting the generators is an automorphism preserving $\tilde{\pi}$ and induces an automorphism of $\tilde{H}(n,2)$ fixing γ. It is clear that all such automorphisms preserve the pair $\{\delta, \tau\delta\}$ and that the automorphism corresponding to a transposition cannot fix δ (because $\tilde{H}(n,2)$ has no 4-cycle).

Now, since the group of all automorphisms of $H(n,2)$ fixing a given point is the full symmetric group and any automorphism of $\tilde{H}(n,2)$ induces an automorphism of $H(n,2)$ by (c), (d) readily follows.

Suppose Γ is the collinearity graph of a rectangular near $2d$-gon of order $(2, t; t_2, \ldots, t_{d-1})$ and fix a point ω in Γ. Let the $1+t$ lines through ω be labelled $1, 2, \ldots, 1+t$ and the two points in $\Gamma_1(\omega)$ of line j be labelled j_0 and j_1. Thus each point in $\Gamma_1(\omega)$ is uniquely determined by its label. Points of $\Gamma_1(\omega)$ will be identified with their labels.

We shall also attach labels to points in $\Gamma_d(\omega)$. Let γ be such a point. Label γ by the vector in $\mathbf{F}_2^{1+t}$ whose jth coordinate $(1 \leqslant j \leqslant 1+t)$ is 0, 1 according as j_0 or j_1 is the nearest point on line j through ω. Two points of $\Gamma_d(\omega)$ may have the same label. Nevertheless, this labelling is useful as is indicated by the lemma below.

Denote by $H^0(n, 2)$ for n odd the graph on the 2^{n-1} vectors in $\mathbf{F}_2^n$ of even weight (the *weight* of a vector being the number of nonzero coordinates) in which two points are adjacent whenever their vector sum has weight $n-1$.

LEMMA 3. *Suppose Γ is the collinearity graph of a generalized $2d$-gon of order $(2, 2^a)$ and ω is a point of Γ. Then the labelling makes each connected component Y of $\Gamma_d(\omega)$ into an m_Y-covering of a graph isomorphic with $H^0(2^a+1, 2)$, for some integer m_Y. The sum $\sum m_Y$, extended over all connected components, is equal to $2^{(ad-a+d-2^a)}$.*

PROOF. If two points of $\Gamma_d(\omega)$ are adjacent in that graph, their labellings differ by precisely 2^a coordinates; furthermore, the places in which the labels of two distinct neighbours of a given point γ coincide with the label of γ are not the same. These facts are consequences of the axioms of generalized $2d$-gons.

In order to prove the first assertion for a given component Y of $\Gamma_d(\omega)$, we may assume, without loss of generality, that Y has a point labelled $00, \ldots, 0$ and then the above observations and the fact that Y and $H^0(2^a+1, 2)$ have the same valency 2^a+1 readily imply that the labelling of Y is a covering of $H^0(2^a+1, 2)$. The second assertion follows from a straightforward enumeration.

5. PROOF OF THEOREM 1

Here, Γ is the collinearity graph of a generalized hexagon of order $(2, 2)$. Fix a point ω of Γ. The idea is to show first that $\Gamma_3(\omega)$ is one of two possible graphs and then that $\Gamma_3(\omega)$ determines Γ uniquely.

The points of $\Gamma_1(\omega)$ and $\Gamma_3(\omega)$ are labelled as in Section 4, and we also label the edges of $\Gamma_3(\omega)$ as follows: such an edge $\{\gamma, \delta\}$ is labelled i_j whenever $\gamma * \delta \in \Gamma_1(i_j)$, where $i_j \in \Gamma_1(\omega)$. In this case, we say that $\{\gamma, \delta\}$ is of *type i*. Thus the type of an edge in $\Gamma_3(\omega)$ is the line through ω to which the edge is nearest.

The proof consists of 12 steps and a proposition.

STEP 1. *If $\{\alpha, \beta\}$ and $\{\gamma, \delta\}$ are distinct edges in $\Gamma_3(\omega)$ with the same label, then α, γ have mutual distance $\geqslant 2$ inside $\Gamma_3(\omega)$.*

Clearly, the two edges are not adjacent. Moreover, $\gamma \in \Gamma_1(\alpha)$ would imply the existence of the pentagon $\alpha\gamma(\gamma * \delta)i_j(\alpha * \beta)\alpha$ where i_j is the common label of $\{\alpha, \beta\}$ and $\{\gamma, \delta\}$.

STEP 2. *If in the path $\alpha\beta\gamma\delta\varepsilon$ of $\Gamma_3(\omega)$ without repetition, the edges $\{\alpha, \beta\}$ and $\{\delta, \varepsilon\}$ are of the same type, they have the same label.*

Let j_0 be the label of $\{\alpha, \beta\}$ and let $\zeta \in \Gamma_1(\gamma) \cap \Gamma_3(\omega)$ be distinct from β, δ. Then $\{\gamma, \zeta\}$ has label j_1 by step 1, so $\{\delta, \varepsilon\}$ has label j_0, again by step 1.

STEP 3. *Set $\{i, k, l\} = \{1, 2, 3\}$. In $\Gamma_3(\omega)$, a path $\alpha\beta\gamma\delta\varepsilon$ without repetition whose edges have types i, k, l, i respectively, extends to a single hexagon inside $\Gamma_3(\omega)$ whose edge types are either i, k, l, i, k, l or i, k, l, i, l, k.*

In Γ there is a point $\eta \in \alpha\beta$ at distance 2 from ε. Let $\zeta \in \Gamma$ be such that $\{\zeta\} = \Gamma_1(\varepsilon) \cap \Gamma_1(\eta)$. Clearly $\eta \neq \beta$. If we had $\eta \neq \alpha$, then $\eta\zeta\varepsilon(\delta * \varepsilon)i_j\eta$, where i_j is the common label of $\{\alpha, \beta\}$ and $\{\delta, \varepsilon\}$ (see step 2), would be a pentagon. Thus $\eta = \alpha$. Suppose $\zeta \in \Gamma_2(\omega)$. Then $\{\alpha, \alpha * \zeta\}$ and $\{\varepsilon, \varepsilon * \zeta\}$ are edges of $\Gamma_3(\omega)$ with the same label. Without loss of generality we may take this label to be k_0. By step 2, the label of $\{\beta, \gamma\}$ must then be k_0, but by (1) it must be k_1. This is a contradiction, whence $\zeta \in \Gamma_3(\omega)$. Step (3) now readily follows.

STEP 4. *In $\Gamma_3(\omega)$ each hexagon has two edges of each type* 1, 2, 3.

By step 1, no type could occur three times.

A hexagon in $\Gamma_3(\omega)$ is said to be *periodic* (with respect to ω) if its edges have types i, j, k, i, j, k and *aperiodic* (with respect to ω) otherwise, i.e., if its edges have types i, j, k, i, k, j.

STEP 5. *A path $\alpha_1\alpha_2\alpha_3$ in $\Gamma_3(\omega)$ of length 2 contained in a periodic hexagon is not contained in a second hexagon.*

Let $\alpha_1\alpha_2\alpha_3\alpha_4\alpha_5\alpha_6\alpha_1$ be the periodic hexagon, with edges of types 1, 2, 3, 1, 2, 3 say. Without loss of generality, we may assume a second hexagon containing $\alpha_1\alpha_2\alpha_3$ to be either $\alpha_1\alpha_2\alpha_3\alpha_4\beta_1\beta_2\alpha_1$ or $\alpha_1\alpha_2\alpha_3\beta_1\beta_2\beta_3\alpha_1$ with $\beta_i \neq \alpha_j$. Application of step 4 yields a contradiction in the first case and determines the edge types of the second hexagon in the latter case, showing that there are two incident edges both of type 3, a contradiction.

STEP 6. *Let Ω be a connected component of $\Gamma_3(\omega)$ containing a periodic hexagon. Then every edge of Ω is in exactly two hexagons and every hexagon in Ω is periodic.*

Let $\alpha_1\alpha_2\alpha_3\alpha_4\alpha_5\alpha_6\alpha_1$ be a hexagon in Ω with edge types i, j, k, i, j, k. From α_2 there starts a path $\alpha_2\beta_1\beta_2\beta_3$ without repetitions and of edge types k, j, i. Apply step 3 to obtain a hexagon $\alpha_1\alpha_2\beta_1\beta_2\beta_3\beta_4\alpha_1$. Note that $\alpha_6 \neq \beta_4$ in view of step 5 and that this hexagon is periodic. Now any hexagon distinct from $\alpha_1\alpha_2\alpha_3\alpha_4\alpha_5\alpha_6\alpha_1$ but containing α_1, α_2 must contain $\beta_1\alpha_2\alpha_1\beta_4$, so coincides with the second hexagon, by step 5.

So far, we have shown that any edge contained in a periodic hexagon in Ω belongs to exactly one other hexagon, which is periodic. As Ω is connected, step 6 readily follows.

STEP 7. *If Ω is a connected component of $\Gamma_3(\omega)$ containing a periodic hexagon, then any path in Ω with edge types i, j, i, j, i, j, i, j (i, j distinct) is an octagon. Moreover, the first and last edges of any path of type k, i, j, i, j, k (i, j, k distinct) span distinct lines meeting in $\Gamma_2(\omega)$.*

Set $\{i, j, k\} = \{1, 2, 3\}$ and let $\gamma_0\gamma_1\gamma_2\gamma_3\gamma_4\gamma_5\gamma_6$ be a path in Ω with edge types k, i, j, i, j, k. Then, clearly, all γ_i ($0 \leqslant i \leqslant 6$) are distinct. We show that $\gamma_0 * \gamma_1 = \gamma_5 * \gamma_6$. For, letting $\beta \in \Gamma_1(\gamma_3) \cap \Gamma_3(\omega)$ be such that $\{\beta, \gamma_3\}$ has type k, we get $\beta * \gamma_3 \in \Gamma_2(\gamma_0 * \gamma_1) \cap \Gamma_2(\gamma_5 * \gamma_6)$. Since $\{\gamma_5, \gamma_6\}$, $\{\gamma_0, \gamma_1\}$, $\{\beta, \gamma_3\}$ have the same label, say k_0, the points $\gamma_0 * \gamma_1$ and $\gamma_5 * \gamma_6$ must be on the same line through k_0. So if $\gamma_0 * \gamma_1 \neq \gamma_5 * \gamma_6$, then $\gamma_4\gamma_5(\gamma_5 * \gamma_6)(\gamma_0 * \gamma_1)$ and $\gamma_4\gamma_3\gamma_2\gamma_1$ are 3-paths from γ_4 to points of $\gamma_0\gamma_1$. Therefore, there exists a unique $\alpha \in \Gamma_1(\gamma_4) \cap \Gamma_1(\gamma_0)$, and $\alpha\gamma_4$ contains an edge of type k. But α is not in $\Gamma_3(\omega)$, for otherwise $\alpha\gamma_4\gamma_3\gamma_2\gamma_1\gamma_0\alpha$ would be an aperiodic hexagon in Ω, contradicting step 6. Hence $\alpha \in \Gamma_2(\omega)$. Now $\alpha\gamma_0$ contains an edge of $\Gamma_3(\omega)$ of type k, so α belongs to the line $\gamma_0\gamma_1$ leading to a pentagon $\alpha\gamma_1\gamma_2\gamma_3\gamma_4\alpha$, a contradiction. This proves that $\gamma_0 * \gamma_1$ and $\gamma_5 * \gamma_6$ coincide, and settles the last statement of step 7. Now if $\gamma_0\gamma_1\delta_2\delta_3\delta_4\delta_5\delta_6$ is a path in Ω with edge

types k, j, i, j, i, k, the statement already proven shows that $\gamma_5 * \gamma_6 = \delta_5 * \delta_6$, hence that $\gamma_5\gamma_6$ and $\delta_5\delta_6$ coincide. One cannot have $\gamma_5 = \delta_6$, otherwise steps 6 and 3, applied to the path $\gamma_3\gamma_4\gamma_5\delta_5\delta_4$, would imply that $\{\delta_3, \gamma_3\}$ is an edge of $\Gamma_3(\omega)$, yielding a pentagon $\gamma_1\gamma_2\gamma_3\delta_3\delta_2\gamma_1$. Thus, $\gamma_5 = \delta_5$ and $\delta_5\delta_4\delta_3\delta_2\gamma_1\gamma_2\gamma_3\gamma_4\gamma_5$ is a closed path of Ω with edge types i, j, i, j, i, j, i, j. This proves step 7.

If $\Gamma_3(\omega)$ contains a periodic hexagon, then ω will be called a *periodic point* of Γ, otherwise ω will be called an *aperiodic point.*

STEP 8. *Let ω be a periodic point of Γ. If Γ' is a generalized hexagon of order $(2, 2)$ containing a periodic point ω', there is an isomorphism $\Gamma \to \Gamma'$ mapping ω to ω'. Moreover, the group of automorphisms of Γ fixing ω is transitive on every set $\Gamma_i(\omega)$ $(0 \leq i \leq 3)$, and $\Gamma_3(\omega)$ is connected.*

Let Ω be a connected component of $\Gamma_3(\omega)$ containing a periodic hexagon. As in the proof of Lemma 2, but now with $n = 3$, let F be the group with generators r_1, r_2, r_3 and relations $r_1^2 = r_2^2 = r_3^2 = 1$, and identify the paths on Ω with origin a fixed point o with the elements of F. In view of steps 3 and 6 any path of the form $gr_ir_jr_kr_ir_jr_kg^{-1}$ with i, j, k distinct and $g \in F$, is an element of $\pi_1(\Omega)$, the fundamental group of Ω with respect to o. Let $\tilde{\pi}$ be the subgroup of $\pi_1(\Omega)$ generated by all these elements and let $\tilde{\Omega}$ be the cover of Ω whose fundamental group is $\tilde{\pi}$. Then the graph $\tilde{\Omega}$ can be described as the collection of points of a hexagonic lattice in the Euclidean plane whose edges are the pairs of points at minimal distance (cf. Figure 2, but disregard labels). The three edge types correspond to the three 'parallel classes' of edges. Let (i, j, k) be a cyclic permutation of $(1, 2, 3)$. The map sending $\gamma \in \tilde{\Omega}$ to the end point of the 2-path starting at γ with edge types i, j if γ has even distance to o and with edge types j, i otherwise is a type preserving automorphism of $\tilde{\Omega}$; call it v_k. It corresponds to a translation of the plane; in fact, these maps v_k $(k = 1, 2, 3)$ generate the full group T of translations stabilizing $\tilde{\Omega}$. Together with the automorphisms of $\tilde{\Omega}$ induced by central symmetries with respect to the center of either a hexagon or an edge, these translations form the full group $\tilde{G}$ of type preserving automorphisms of $\tilde{\Omega}$. It is regular on the vertices. Now Ω is the quotient of $\tilde{\Omega}$ by a subgroup, G say, of $\tilde{G}$. But G cannot contain a central symmetry, for otherwise Ω would contain either a triangle or a loop. Thus $G \subset T$. Due to step 7, the elements $4v_i$ $(i = 1, 2, 3)$ belong to G and the translations $2v_i$ cannot belong to G. Hence $[T : G]$ divides 16 and G contains no subgroup strictly larger than the one generated by $4v_1$, $4v_2$, $4v_3$. It follows that G coincides with the latter and that the number of vertices of Ω is $[\tilde{G} : G] = 2[T : G] = 32$. In particular, $\Omega = \Gamma_3(\omega)$ and the group $\tilde{G}/G$ acts on Ω as a regular group of type preserving automorphisms.

By the second statement of step 7, the incidence system $Y = Y_{\geq 2}(\omega)$ as defined in Section 3 can be uniquely reconstructed from Ω. The first part of step 8 is therefore a direct consequence of Lemma 1. The second statement follows by taking the automorphisms of Ω (not necessarily type preserving) induced from those of $\tilde{\Omega}$, extending them to automorphisms of Y and applying Lemma 1 once more.

STEP 9. *If ω is aperiodic, $\Gamma_3(\omega)$ is the union of two connected components each of which is isomorphic to $\tilde{H}(3, 2)$* (cf. Lemma 2).

Due to steps 6 and 8, every hexagon in $\Gamma_3(\omega)$ is aperiodic. Let Ω be a connected component of $\Gamma_3(\omega)$, and fix a point o in Ω. Define F as in the proof of Lemma 2 and identify the paths originating in o again by the elements of F. In view of step 3 and the aperiodicity of all hexagons in Ω, any path of the form $gr_jr_kr_ir_jr_ir_kg^{-1}$ with i, j, k distinct and $g \in F$ is an element of $\pi_1(\Omega)$ (identified with a subgroup of F). According to Lemma

2(b) this implies that $\pi_1(\tilde{H}(3,2))$ is contained in $\pi_1(\Omega)$. It is easily seen that the sixteen elements

$$p(a,b,c,d) = r_1^a r_2^b r_3^c (r_1 r_2 r_1 r_2)^d, \qquad a,b,c,d \in \{0,1\},$$

form a set of coset representatives of $\pi_1(\tilde{H}(3,2))$ in F. If $d=0$ or if $b=c=0$, $p(a,b,c,d)$ represents a path of length strictly smaller than 6. If $d=c=1$ (resp. if $d=b=1$ and $c=0$), the coset of $p(a,b,c,d)$ also contains

$$r_1^a r_2^{b+1} r_3 r_2 \quad (\text{resp. } r_1^{a+1} r_2 r_1),$$

which again represents a path of length <6. Consequently, none of the $p(a,b,c,d)$, except $p(0,0,0,0)$, can belong to $\pi_1(\Omega)$, and we have $\pi_1(\Omega) = \pi_1(\tilde{H}(3,2))$, hence step 9.

Let $\Gamma_3(\omega)$ be as in step 9. For $\gamma \in \Gamma_3(\omega)$, there is a unique point at distance 4 to γ within $\Gamma_3(\omega)$ with the same label as γ. Denote this point by γ^σ.

STEP 10. *Let ω be aperiodic and let $\gamma \in \Gamma_3(\omega)$. If $\delta \in \Gamma_3(\omega) \cap \Gamma_1(\gamma)$, then $\gamma * \delta$ is collinear with $\gamma^\sigma * \delta^\sigma$ and σ can be extended to an involutory automorphism of Γ fixing $\Gamma_0(\omega) \cup \Gamma_1(\omega)$ pointwise.*

Any $\delta \in \Gamma_3(\omega) \cap \Gamma_1(\gamma)$ has distance 3 to γ^σ inside $\Gamma_3(\omega)$ and is therefore in $\Gamma_3(\gamma^\sigma)$. However, among the three paths of length 3 joining δ and γ^σ, only two are inside $\Gamma_3(\omega)$; therefore, the third one must be $\delta(\gamma * \delta)(\gamma^\sigma * \delta^\sigma)\gamma^\sigma$. This proves the first statement. Extending σ to an automorphism of $Y_{\geq 2}(\omega)$, defined as in Section 3, by letting α^σ, for $\alpha \in \Gamma_2(\omega)$, be the unique point in $\Gamma_2(\omega) \cap \Gamma_1(\alpha)$, and applying Lemma 1, we obtain the second statement.

STEP 11. *Let ω be an aperiodic point. If five points of a hexagon in Γ belong to $\Gamma_3(\omega)$, then so does the sixth.*

Let $\gamma\gamma_1\delta_1\delta$ be a 3-path inside $\Gamma_3(\omega)$. It has edge types either i,j,i or i,j,k with i,j,k distinct. In the latter case, applying step 3 to the path $\gamma\gamma_1\delta_1\delta$ (resp. $\delta\delta_1\gamma_1\gamma$) extended by an edge of type i (resp. k) provides a 3-path from γ to δ with edge types j,k,i (resp. k,i,j); thus, in that case, all three 3-paths joining γ and δ are contained in $\Gamma_3(\omega)$. In the former case, application of step 3 to the path with edge types j,i,k,j originating from γ_1 leads to a second 3-path from γ to δ with edge types k,j,k. Now, the third 3-path from γ to δ must pass through points on the lines spanned by the edges of type j with extremities γ, δ respectively and hence has no other points but γ and δ in $\Gamma_3(\omega)$. The conclusion is that a hexagon containing γ and δ has either 4 or 6 points in $\Gamma_3(\omega)$. This establishes step 11.

STEP 12. *If ω is aperiodic, then every line through ω is a periodic point of the dual Γ^* of Γ.*

Let Ω be a connected component of $\Gamma_3(\omega)$ and let ω^* denote the line labelled 1 through ω. Let γ, δ be adjacent points of $\Gamma_3(\omega)$ such that $\{\gamma, \delta\}$ has type 3. Then by step 10 there is a line e of Γ on $\gamma * \delta$ and $\gamma^\sigma * \delta^\sigma$. Since e has a point on the line 3 through ω, we have $e \in \Gamma_2^*(\omega^*)$. Now δ, γ^σ have distance 3 realized by two distinct paths of Ω, say $\delta\alpha_1\beta_1\gamma^\sigma$ and $\delta\alpha_2\beta_2\gamma^\sigma$, respectively. Note that $\delta\alpha_1\beta_1\gamma^\sigma\beta_2\alpha_2\delta$ is a hexagon in Ω. Since the type of $\{\gamma, \delta\}$ is 3, we may assume that the type of $\{\delta, \alpha_i\}$ is i for $i = 1, 2$. Since the hexagon is not periodic by the hypothesis, it follows from step 4 and the fact that $\{\gamma^\sigma, \delta^\sigma\}$ has type 3, that $\{\gamma^\sigma, \beta_i\}$ has type i for $i = 1, 2$, and that $\{\alpha_2, \beta_2\}$ has type 3. This means that the lines $\delta\alpha_2$, $\alpha_2\beta_2$, $\beta_2\gamma^\sigma$ of Γ are points of $\Gamma_3^*(\omega^*)$. In particular, the closed line-path e, $\gamma\delta$, $\delta\alpha_2$, $\alpha_2\beta_2$, $\beta_2\gamma^\sigma$, $\gamma^\sigma\delta^\sigma$, e constitutes a hexagon of Γ^* having exactly five points in

$\Gamma_3^*(\omega^*)$, namely all but e. Thanks to step 11 it follows that $\Gamma_3^*(\omega^*)$ contains a periodic hexagon.

PROPOSITION. *In a generalized hexagon of order* $(2, 2)$ *either all points are periodic or all points are aperiodic. Each instance corresponds to a single isomorphism class, and these classes are the duals of each other.*

PROOF. Suppose Γ contains two periodic points ω and ω'. Let i be the distance from ω to ω'. Since by step 8 the group of automorphisms fixing ω is transitive on $\Gamma_i(\omega)$, the graph Γ contains a periodic point adjacent to ω'. But then by the same argument, all neighbours of ω' are periodic. By induction with respect to the length of a path from ω' and in view of the connectedness of Γ, it follows that all points of Γ are periodic. Consequently, if Γ contains a periodic point γ as well as aperiodic points, then all points except for γ are aperiodic. On the other hand, if σ is the involutory automorphism of step 10 defined with respect to a point of $\Gamma_3(\gamma)$, then γ^σ is a periodic point distinct from γ, which is absurd. Hence the first assertion.

In view of steps 8 and 12 there are at most two isomorphism classes of generalized hexagons of order $(2, 2)$. Since the classical generalized hexagon of this order is not isomorphic to its dual (cf. e.g. [8, 5.9]), there are exactly two isomorphism classes (one could of course also derive this fact by verifying that any line through a periodic point is an aperiodic point of the dual). These two classes are each other's duals, due to step 12. The proposition is proved.

Theorem 1 is a direct consequence of the proposition.

REMARKS. Let Γ, ω be as before and suppose ω is aperiodic. The automorphism σ of step 10 is used by Timmesfeld in [6]. It clearly is central in the group of automorphisms fixing ω. Since the full automorphism group is transitive (consider the subgroup generated by all σ for ω running through the points of Γ to establish this), and Γ has odd cardinality (namely 63), it follows that σ is contained in the centre of a Sylow 2-subgroup of the full automorphism group. This fact may be used to identify Γ as the dual of the classical generalized hexagon. Thus, the classical generalized hexagon is the one with periodic points. With the data provided by the steps of the proof it is not hard to explicitly describe the two generalized hexagons. We have done so in Figures 1 and 2.

From step 10 it follows that if ω is an aperiodic point and if $\gamma \in \Gamma_3(\omega)$, then $\gamma\tilde{\omega} = \{\gamma^\sigma\}$. This implies that the regulus condition holds for the dual of the classical hexagon. But it is easily shown that, for hexagons of order $(2, 2)$, the regulus condition is self-dual; therefore it also holds for the classical one.

6. PROOF OF THEOREM 2

Here, Γ is the collinearity graph of a generalized hexagon of order $(2, 8)$. Fix $\omega \in \Gamma$ and label the points of $\Gamma_1(\omega)$ and $\Gamma_3(\omega)$ as above. By [4], any generalized hexagon of order $(2, 8)$ in which the regulus condition holds, is the ${}^3D_4(2)$-hexagon. We shall therefore content ourselves with the following proof of the regulus condition, in two steps.

STEP 1. *The graph* $\Gamma_3(\omega)$ *is connected. Its labelling is a* 2*-cover of* $H^0(9, 2)$. *For any* $\gamma \in \Gamma_3(\omega)$, *the unique point* γ^σ *of* $\Gamma_3(\omega)$ *which is distinct from* γ *and has the same label belongs to* $\Gamma_3(\gamma)$. *Moreover, if* $\delta \in \Gamma_1(\gamma) \cap \Gamma_3(\omega)$, *then* $\delta \in \Gamma_3(\gamma^\sigma)$.

By Lemma 3, there is a map $\xi \mapsto \bar{\xi}$ of $\Gamma_3(\omega)$ onto $H^0(9, 2)$ which is a 2-cover and which, restricted to any connected component of $\Gamma_3(\omega)$, is isomorphic to the labelling. Let γ, δ

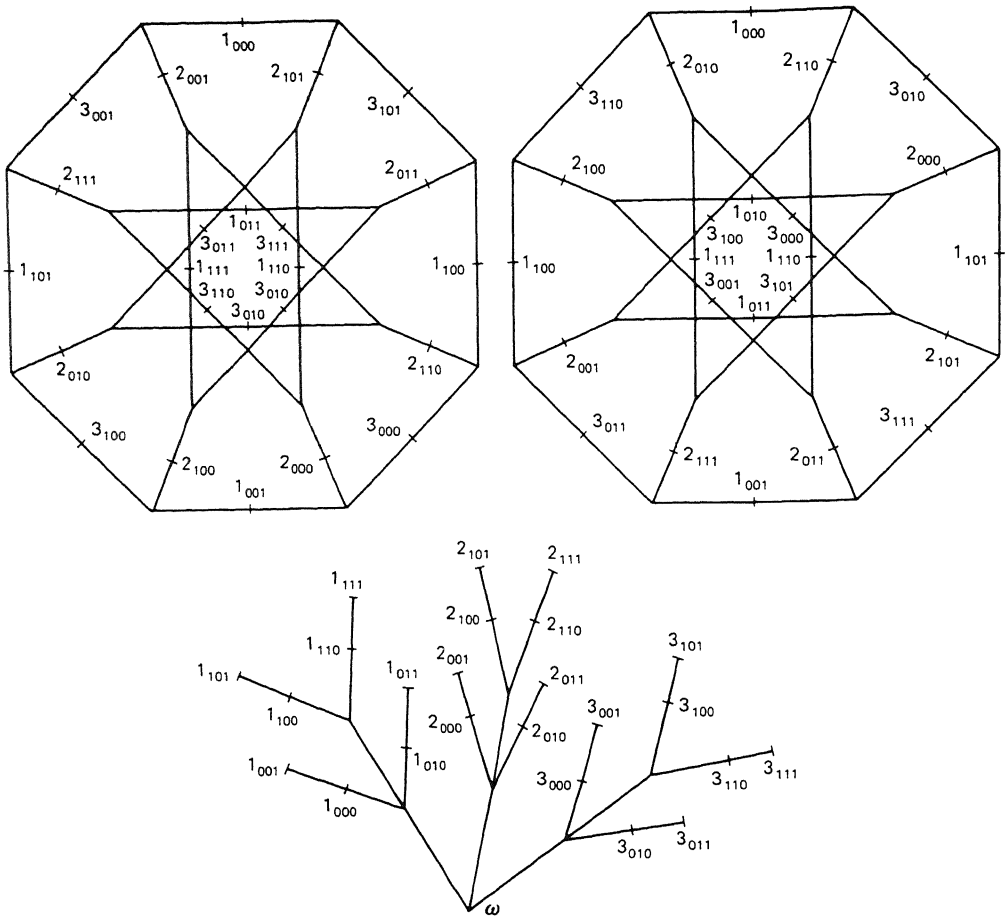

FIGURE 1. The dual of the classical generalized hexagon of order (2, 2). Line segments represent lines of the generalized hexagon. The three vertices labelled i_{jkl} in the picture are to be identified in order to represent a point in $\Gamma_2(\omega) \cap \Gamma_1(i_j)$. Each occurrence of the vertex i_{jkl} provides a line through the point i_{jkl}.

be two adjacent points of $\Gamma_3(\omega)$; their images $\bar{\gamma}$, $\bar{\delta}$ are adjacent. Take a 3-path in $H^0(9, 2)$ from $\bar{\delta}$ to $\bar{\gamma}$ and lift it to a 3-path in $\Gamma_3(\omega)$ originating in δ. The end point of that 3-path has the same label as γ and is the end point of a 4-path starting at γ. Therefore, it must be distinct from γ, whence the first, second and fourth assertions of step 1.

Since γ and γ^σ are not adjacent in $\Gamma_3(\omega)$, their mutual distance is $\geqslant 2$. Suppose there exists $\delta \in \Gamma_1(\gamma) \cap \Gamma_1(\gamma^\sigma)$. Then necessarily $\delta \in \Gamma_2(\omega)$, so that $\gamma * \delta \in \Gamma_3(\omega)$. The latter point, being a neighbour of γ, has distance 3 to γ^σ in $\Gamma_3(\omega)$. Thus $\gamma^\sigma \delta(\delta * \gamma)$ can be completed to a pentagon, which is absurd. It follows that $\gamma^\sigma \notin \Gamma_2(\gamma)$, so that $\gamma^\sigma \in \Gamma_3(\gamma)$, whence the first statement.

STEP 2. *The graph Γ satisfies the regulus condition.*

Let $\{\gamma, \delta\}$ be an edge of $\Gamma_3(\omega)$. Then $\{\gamma^\sigma, \delta^\sigma\}$ is also an edge. The common label of γ and γ^σ coincides in a single place, say the ith, with the label of δ and δ^σ. If j is the common value of those labels in that place, we have $\Gamma_1(\gamma^\sigma * \delta^\sigma) \cap \Gamma_1(\omega) = \{i_j\}$. By step 1 the points $\gamma * \delta$ and $\gamma^\sigma * \delta^\sigma$ are distinct. Suppose they are nonadjacent. Then $\delta^\sigma(\gamma^\sigma * \delta^\sigma)$ $i_j(\gamma * \delta)$ is a 3-path, so that $\delta^\sigma \in \Gamma_3(\gamma * \delta)$. As $\delta^\sigma \in \Gamma_3(\gamma)$ by step 1, we get $\delta^\sigma \in \Gamma_2(\delta)$, in

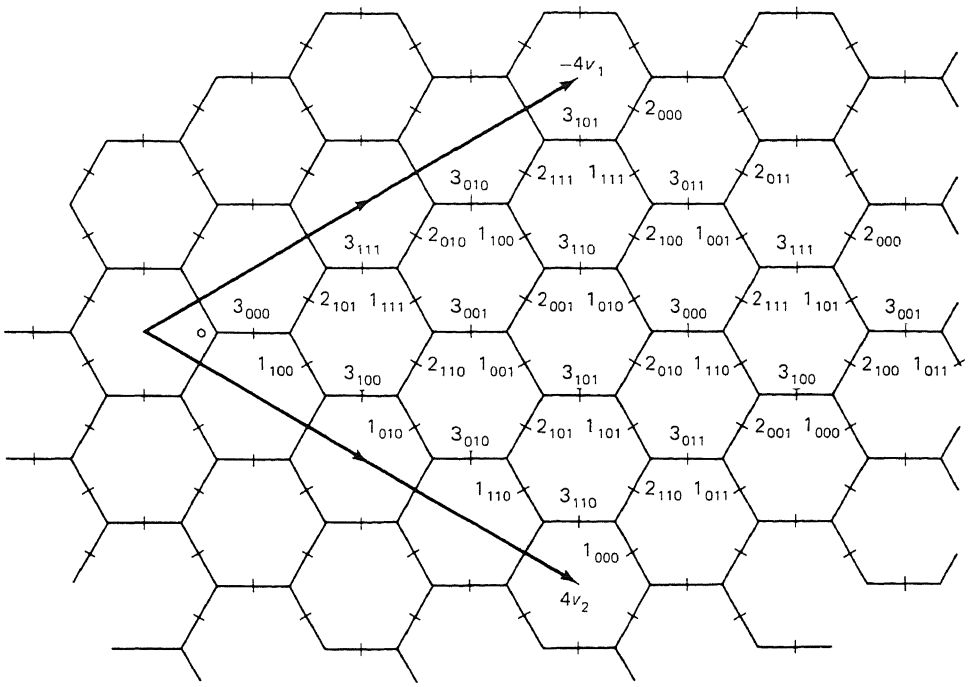

FIGURE 2. The classical generalized hexagon of order (2, 2). Conventions are as in Figure 1, with the exception that the constituent $\Gamma_0(\omega)\cup\Gamma_1(\omega)\cup\Gamma_2(\omega)$ is not included as it is identical to the one in Figure 1.

contradiction with step 1. The conclusion is that $\gamma * \delta$ and $\gamma^\sigma * \delta^\sigma$ must be adjacent and that $i_j = (\gamma * \delta) * (\gamma^\sigma * \delta^\sigma)$. Letting δ range over $\Gamma_1(\gamma)\cap\Gamma_3(\omega)$, we obtain $\gamma^\sigma \in \omega\tilde{\gamma}$ and we are done.

REMARK. Instead of ending the proof by referring to [4], we could also observe that σ can be extended to an involutory automorphism of Γ and apply Timmesfeld's Theorem [6; (3, 3)] to the group generated by the 819 involutions σ obtained by varying ω over the points of Γ.

7. PROOF OF THEOREM 3

Here, Γ is the collinearity graph of a regular near 8-gon of order $(2, 4; 0, 3)$. Fix $\omega \in \Gamma$ and label $\Gamma_1(\omega)$, $\Gamma_4(\omega)$ as above. We proceed in 8 steps.

STEP 1. *The graph $\Gamma_4(\omega)$ is isomorphic to $\tilde{H}(5, 2)$* (cf. Lemma 2).

The labels of two adjacent points γ, δ of $\Gamma_4(\omega)$ differ in exactly one place; they coincide in four places as $\gamma * \delta$ has distance 2 to $1 + t_3 = 4$ points on four distinct lines through ω, and the coordinates corresponding to the fifth line through ω must differ, for otherwise this line through ω would bear a point at distance 3 from γ, δ and this point would be a fifth point of $\Gamma_1(\omega)\cap\Gamma_2(\gamma * \delta)$, in contradiction with $t_3 = 3$. Thus the labelling is a 2-cover of $H(5, 2)$. Thanks to Lemma 2, we are done.

STEP 2. *Put $\Omega_i = \{\gamma \in \Gamma_3(\omega) \mid \Gamma_2(\gamma)\cap\{0_i, 1_j\} = \varnothing\}$ for $1 \leqslant i \leqslant 5$. Then Γ is the disjoint union of the Ω_i and each Ω_i is isomorphic to $\tilde{H}(4, 2)$* (cf. Lemma 2).

The points of Ω_i are labelled by vectors in $\mathbf{F}_2^4$ whose coordinates are indexed by the numbers j $(1 \leqslant j \leqslant 5; j \neq i)$ in such a way that the jth coordinate of the label of $\gamma \in \Omega_i$ is 0, 1 according as $0_j \in \Gamma_2(\gamma)$ or $1_j \in \Gamma_2(\gamma)$. Thus, the label of a point in Ω_i is obtained by deletion of the ith coordinate from the label of either one of its two neighbours in $\Gamma_4(\omega)$. The defining properties of regular near octagons readily imply that any edge $\{\gamma, \delta\}$ with $\gamma \in \Omega_i$ and $\delta \in \Gamma_3(\omega)$ must belong to Ω_i and that the labels of γ and δ differ in all but one coordinate. But for any even n, the graph on $\mathbf{F}_2^n$ whose edges are the pairs of points all of whose coordinates but one are different is isomorphic with the n-cube $H(n, 2)$. Thus, Ω_i is a cover of $H(4, 2)$ without 4-circuits. In particular, it has at least 32 points. But $\Gamma_3(\omega)$ has 160 points in all, so each Ω_i is a 2-cover of $H(4, 2)$, hence the assertion.

Let σ (depending on ω) be the permutation of the set of vertices of Γ defined as follows: if $\gamma \in \Gamma_0(\omega) \cup \Gamma_1(\omega)$, then $\gamma^\sigma = \gamma$; if $\gamma \in \Gamma_2(\omega)$, then $\gamma^\sigma = \gamma * \delta$, where $\omega\delta\gamma$ is the unique 2-path joining ω and γ; finally, if $\gamma \in \Gamma_i(\omega)$ with $i = 3$ or 4, γ^σ is the unique second point of the connected component of γ in $\Gamma_i(\omega)$ whose label coincides with the label of γ.

STEP 3. *For any* $\gamma \in \Gamma_3(\omega)$, *we have* $\gamma^\sigma \in \Gamma_3(\gamma) \cup \Gamma_4(\gamma)$. *Moreover, if* $\delta \in \Gamma_3(\omega) \cap \Gamma_1(\gamma)$, *then* $\delta \in \Gamma_3(\gamma^\sigma)$.

STEP 4. *The graph* Γ *satisfies the regulus condition. Moreover, the map* σ *is an involutory automorphism of* Γ.

Proofs of step 3 and the first part of step 4 are omitted as they run parallel to those of step 1 (last two statements) and step 2 for Theorem 2.

As for the involution σ, we first check that it carries lines in $\Gamma_2(\omega) \cup \Gamma_3(\omega)$ to lines. Suppose L is a line contained in $\Gamma_2(\omega) \cup \Gamma_3(\omega)$. Then there are $\alpha \in \Gamma_3(\omega)$ and $\beta \in \Gamma_2(\omega)$ such that $L = \{\alpha, \beta, \alpha * \beta\}$ with $\alpha * \beta \in \Gamma_3(\omega)$. Without harming generality, we assume that α is in Ω_5 and has label 0000 and that $\alpha * \beta$ has label 0111 in Ω_5. In view of the regulus condition, there is a unique point $\alpha' \in \omega\tilde{\alpha}$ (see Section 2). This point must be in $\Gamma_3(\omega)$, distinct from α, and at distance 2 to each point of $\Gamma_1(\omega) \cap \Gamma_2(\alpha)$. Therefore, α' is contained in Ω_5 and has the same label in Ω_5 as α. Thus, $\alpha' = \alpha^\sigma$. Moreover, if $\varepsilon \in \Gamma_1(\omega)$ is such that $\omega\varepsilon\beta$ is a 2-path, the points $\varepsilon * \beta$ and α^σ are adjacent. But $\varepsilon * \beta = \beta^\sigma (\in \Gamma_2(\omega))$ by definition, so $\{\alpha^\sigma, \beta^\sigma\}$ is an edge. Since $\alpha^\sigma * \beta^\sigma$ is an element of $\Gamma_1(\varepsilon * \beta) \cap \Gamma_1(\alpha^\sigma)$, it must be an element of Ω_5 distinct from $\alpha * \beta$ with label 0111 in Ω_5. Consequently, $\alpha^\sigma * \beta^\sigma = (\alpha * \beta)^\sigma$ and L^σ is the line $\{\alpha^\sigma, \beta^\sigma, \alpha^\sigma * \beta^\sigma\}$.

If $\{\gamma, \delta\}$ is an edge with $\gamma \in \Gamma_3(\omega)$ and $\delta \in \Gamma_4(\omega)$, then there is a unique line on γ^σ containing a point of $\Gamma_4(\omega)$ with the same label as δ. This point cannot be δ, for otherwise γ and γ^σ would be two neighbours of δ in the same connected component of $\Gamma_3(\omega)$. Therefore, δ^σ must be adjacent to γ^σ. Since σ clearly is an automorphism on $\Gamma_4(\omega)$, we conclude that σ is an automorphism of Γ. This settles step 4.

STEP 5. *Let* $\gamma \in \Gamma_4(\omega)$ *have label* 00000. *Then, the two points of* $\Gamma_4(\omega)$ *labelled* 11111 *are at distance* 4 *from* γ. *One of them, call it* δ, *satisfies the relation* $\Gamma_1(\gamma) \cap \Gamma_3(\delta) \subset \Gamma_3(\omega)$, *whereas the other,* δ^σ, *satisfies* $\Gamma_1(\gamma) \cap \Gamma_3(\delta^\sigma) \subset \Gamma_4(\omega)$. *If* $\gamma\gamma_1\gamma_2\gamma_3\delta$ *is any path of length* 4 *joining* γ *and* δ, *one has* $\gamma_1, \gamma_2 \in \Gamma_3(\omega)$ *and* $\gamma_3 \in \Gamma_4(\omega)$.

Let δ provisorily denote any one of the two points of $\Gamma_4(\omega)$ labelled 11111 (the other being δ^σ). We shall repeatedly make use of the fact that

(*) the labels of two points of $\Gamma_4(\omega)$ adjacent with a same point of $\Gamma_3(\omega)$ differ in at most one place.

In particular, $\delta, \delta^\sigma \in \Gamma_3(\gamma) \cup \Gamma_4(\gamma)$. Let $\gamma\gamma_1\gamma_2\gamma_3\delta$ be a path, with possibly $\gamma_1 = \gamma_2$. If $\gamma_1, \gamma_3 \in \Gamma_3(\omega)$, $(*)$ implies that $\gamma_2 \in \Gamma_2(\omega) \cup \Gamma_3(\omega)$. But γ_1, γ_3 are labelled 0000 and 1111, respectively, so their mutual distance in $\Gamma_3(\omega)$ exceeds 2 and we must have $\gamma_2 \in \Gamma_2(\omega)$, in contradiction with the fact that the labels of γ_1 and γ_3 have no coordinate in common. Thus either γ_1 or γ_3 belongs to $\Gamma_4(\omega)$. We assume $\gamma_3 \in \Gamma_4(\omega)$, the reasoning for the other case being similar. Since the labels of γ and δ differ in all 5 coordinates, $(*)$ implies that $\gamma_1, \gamma_2 \in \Gamma_3(\omega)$ and $\gamma_1 \neq \gamma_2$, hence $\delta \in \Gamma_4(\omega)$. We have $\gamma_1 \in \Gamma_3(\delta)$; therefore, there are four minimal paths from γ_1 to δ and, by $(*)$, each one of them starts with an edge in $\Gamma_3(\omega)$ followed by a point of $\Gamma_4(\omega) \cap \Gamma_1(\delta)$. From any such point there start four paths of length 3 to γ each of which consists of its origin [in $\Gamma_4(\omega) \cap \Gamma_1(\delta)$], two vertices belonging to $\Gamma_3(\omega)$ and, finally, γ [again by $(*)$]. As any point of $\Gamma_1(\gamma) \cap \Gamma_3(\delta)$ must belong to such a path, we have $\Gamma_1(\gamma) \cap \Gamma_3(\delta) \subset \Gamma_3(\omega)$, which is the first inclusion of step 5. The above discussion also establishes the last assertion and shows that any path $\gamma\gamma_1\gamma_2$, with $\gamma_1, \gamma_2 \in \Gamma_3(\omega)$, can be completed to a path $\gamma\gamma_1\gamma_2\gamma_3\delta$. Since $|\Gamma_1(\gamma_2) \cap \Gamma_4(\omega)| = 2$, the same path $\gamma\gamma_1\gamma_2$ can certainly not be completed to a path $\gamma_2\gamma_3'\delta^\sigma$; therefore, one cannot have $\gamma_1 \in \Gamma_3(\delta^\sigma)$. Consequently, $\Gamma_1(\gamma) \cap \Gamma_3(\delta^\sigma) \subset \Gamma_4(\omega)$, which finishes the proof of step 5.

STEP 6. *Suppose the labelled graph $\Gamma_4(\omega)$ is given. Then:*
(a) *if $\gamma\gamma_1$ is an edge of Γ, with $\gamma \in \Gamma_4(\omega)$ and $\gamma_1 \in \Omega_i$, the third point $\gamma * \gamma_1$ of the line it spans is uniquely determined by γ, the index i and the label of γ_1;*
(b) *if ζ and ζ' are two points of Ω_i at mutual distance 2 inside Ω_i, the (unique) edge of $\Gamma_4(\omega)$ collinear with ζ' is uniquely determined by the edge collinear with ζ, the index i and the label of ζ'.*

(a) Clearly, $\gamma * \gamma_1$ is the unique point of $\Gamma_4(\omega)$ connected with γ and whose label coincides with that of γ except in the ith place.

(b) Let $\zeta\gamma_1\zeta'$ be a path inside Ω_i. Assume without loss of generality, that $i = 5$ and that the labels of ζ, γ_1 and ζ' are 0111, 0000 and 1011, respectively. Let ξ be the point of $\Gamma_4(\omega)$ labelled 01111 and connected with ζ (by hypothesis, that point is known). Denote by ξ' the point of $\Gamma_4(\omega)$ labelled 10111 and connected with ζ'. Now, if δ (resp. δ_1) is the (uniquely determined) point of $\Gamma_4(\omega)$ labelled 11111 and connected with ξ (resp. with ξ'), it follows from step 5 that we cannot have $\delta_1 = \delta^\sigma$: just take for γ the point of $\Gamma_4(\omega)$ labelled 00000 and connected with γ_1. Thus $\delta = \delta_1$, and ξ' is the unique point of $\Gamma_4(\omega)$ with label 10111 and such that $\xi\delta\xi'$ is a path. Our assertion follows, in view of (a).

STEP 7. *The graph $\Gamma_3(\omega) \cup \Gamma_4(\omega) = (\bigcup \Omega_i) \cup \Gamma_4(\omega)$ is unique up to label-preserving isomorphism and possibly a transposition of the two first types (or any preassigned odd permutation of the types).*

The labelled graphs $\Gamma_4(\omega)$ and Ω_i are unique by steps 2 and 3. Let $\gamma, \delta \in \Gamma_4(\omega)$ have labels 00000, 11111, respectively, and satisfy the condition of step 5. Up to label-preserving isomorphism, there are *two distinct choices* for the pair (γ, δ), but in view of Lemma 2(d) one can pass from one to the other by an automorphism of the graph $\Gamma_4(\omega)$ inducing any preassigned odd permutation of the types. Now, supposing γ and δ given, we shall show—and that will prove our contention—that the graph $\Gamma_4(\omega) \cup \Omega_i$ is unique up to the label preserving automorphism of Ω_i [extended by the identity on $\Gamma_4(\omega)$]. We assume, without loss of generality, that $i = 5$. Let γ_3 be the point of $\Gamma_4(\omega)$ labelled 01111 and connected with δ. By the assumption made on γ, δ, there exists an edge $\gamma_1\gamma_2$ of Ω_5 whose vertices carry the labels 0000 and 0111 and are connected with γ and γ_3, respectively. Applying step 6 repeatedly, one sees that this edge uniquely determines the graph structure of $\Gamma_4(\omega) \cup \Omega_5$. Now, our assertion follows from the fact that the two edges of Ω_5 with the given labels are permuted by a label-preserving automorphism of Ω_5.

STEP 8. *The incidence system $Y_{\geq 2}(\omega)$ defined as in Section 3 can be uniquely reconstructed from the graph $\Gamma_{\geq 3}(\omega)$ and its partition into $\Gamma_3(\omega)$ and $\Gamma_4(\omega)$.*

To prove this, we need only determine which 4-tuples of edges in $\Gamma_3(\omega)$ span lines meeting in the same point of $\Gamma_2(\omega)$.

Let $\{\gamma, \delta\}$ be an edge of $\Gamma_3(\omega)$. Then $\gamma, \delta \in \Omega_k$ for some k $(1 \leq k \leq 5)$, say $k = 5$. Assume, without loss of generality, that the label of γ is 0000 and that the label of δ is 1110. There are lines $\{\gamma, \gamma^1, \gamma^2\}$ and $\{\delta, \delta^1, \delta^2\}$ with $\gamma^1, \gamma^2, \delta^1, \delta^2 \in \Gamma_4(\omega)$ such that γ^1 has label 00000, γ^2 has label 00001, δ^1 has label 11100, and δ^2 has label 11101. Let $i = 1, 2$. There are four 3-paths from γ^i to δ^i, one of them is $\gamma^i\gamma\delta\delta^i$ and the others are three distinct paths inside $\Gamma_4(\omega)$.

Upon permuting, if necessary, the two first places of the labelling, we may assume that the labels of the points of the 3-paths joining γ^1 and δ^1 inside $\Gamma_4(\omega)$ are

$$00000 \quad \begin{bmatrix} 00100 & 10100 \\ 10000 & 11000 \\ 01000 & 01100 \end{bmatrix} \quad 11100. \tag{L}$$

We denote the corresponding paths by $\gamma^1\gamma_j^1\delta_j^1\delta^1$, where $j = 1, 2, 3$ is the number of the row in (L). Now the labels of the 3-paths joining γ^2 and δ^2 inside $\Gamma_4(\omega)$ cannot be anything but

$$00001 \quad \begin{bmatrix} 01001 & 11001 \\ 00101 & 01101 \\ 10001 & 10101 \end{bmatrix} \quad 11101,$$

and again, we denote the corresponding paths by $\gamma^2\gamma_j^2\delta_j^2\delta^2$. Observe that $\gamma_j^1 * \delta_j^1$ and $\gamma_j^2 * \delta_j^2$ belong to Ω_j and that their labels in Ω_j differ in exactly three places.

By the regulus condition, there exists a point ξ^i $(i = 1, 2)$ connected with $\gamma * \delta$ and $\gamma_j^i * \delta_j^i$ for all j. That point cannot belong to $\Gamma_4(\omega)$ (because it is connected to $\gamma * \delta$) nor to $\Gamma_3(\omega)$ (because no point of $\Gamma_3(\omega)$ can be connected simultaneously to points of Ω_1, Ω_2 and Ω_3). Therefore, ξ^i belongs to $\Gamma_2(\omega)$. Being connected with $\gamma * \delta$, the point ξ^i coincides with $(\gamma * \delta)^\sigma$. Thus, $(\gamma * \delta)^\sigma$ is connected with $\gamma_j^i * \delta_j^i$ for $i = 1, 2$. In particular, $\gamma_j^1 * \delta_j^1$ and $\gamma_j^2 * \delta_j^2$ have distance at most 2 to each other. As their labels in Ω_j differ in exactly three places, they are adjacent (for otherwise, they would have distance 3 in Ω_j and distance at most 2 in Γ, a contradiction with the non-existence of circuits of length 4 or 5). Consequently, $(\gamma * \delta)^\sigma = (\gamma_j^1 * \delta_j^1) * (\gamma_j^2 * \delta_j^2)$, and $\gamma * \delta = (\gamma_j^1 * \delta_j^1)^\sigma * (\gamma_j^2 * \delta_j^2)^\sigma$. Assertion (8) follows.

Theorem 3 is a direct consequence of steps 7 and 8 and Lemma 1.

REMARK. The Hall–Janko group arises as the group of automorphisms generated by the 315 involutions σ for ω ranging over the points of Γ.

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